OPTIMAL ACTUATOR/SENSOR PLACEMENT FOR NONLINEAR CONTROL OF THE KURAMOTO-SIVASHINSKY EQUATION

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KURAMOTO-SIVASHINSKY EQUATION

- Kuramoto-Sivashinsky equation with distributed control:

$$\frac{\partial U}{\partial t} = -\nu \frac{\partial^4 U}{\partial z^4} - \frac{\partial^2 U}{\partial z^2} - U \frac{\partial U}{\partial z} + \sum_{i=1}^{l} b_i u_i(t)$$

- Boundary conditions: $\frac{\partial^j U}{\partial z^j}(-\pi, t) = \frac{\partial^j U}{\partial z^j}(+\pi, t), \quad j = 0, \ldots, 3$

- The stability of steady-state $U(z,t) = 0$ depends on ν.

INTRODUCTION

- Important systems described by the Kuramoto-Sivashinsky equation:
  - ◇ Falling liquid films.
  - ◇ Unstable flame fronts.
  - ◇ Interfacial instabilities between two viscous fluids.
- Feedback control of the Kuramoto-Sivashinsky equation:
  - ◇ Finite-dimensional output feedback controller design based on Galerkin's method (e.g., Armaou and Christofides, CES and Physica D, 2000).
  - ◇ Global stabilization of the zero solution of the KSE (e.g., Christofides and Armaou, SCL, 2000).
  - ◇ Boundary control (e.g., Liu and Krstic, NA, 2001).
- Optimal actuator/sensor placement for the Kuramoto-Sivashinsky equation?

BACKGROUND ON ACTUATOR/SENSOR PLACEMENT

- Optimal actuator placement for linear controllers and PDE models.
  - ◇ Controllability measures (e.g., Arbel, 1981).
  - ◇ Optimal controller gain/actuator location to minimize cost on system response and control action (e.g., Rao *et al*, AIAA J., 1991).
- Optimal sensor placement for linear estimators and PDE models.
  - ◇ Observability measures (e.g., Yu and Seinfeld, IJC. 1973; Waldraff *et al*, JPC, 1998).
  - ◇ Minimum estimation error under worst measurement noise (e.g., Kumar and Seinfeld, IEEE TAC, 1978; Morari and O'Dowd, Automatica 1980).

    Review paper: Kubrusly and Malebranche, Automatica, 1985.
- Optimal actuators/sensors placement for nonlinear dissipative PDE systems (Antoniades and Christofides, C&CE, 2000; CES, 2001; C&CE, 2002).

PRESENT WORK

(Lou and Christofides, IEEE CST, 2002)

- Optimal actuator/sensor placement for nonlinear control of the Kuramoto-Sivashinsky equation.

  ◇ Order reduction using linear/nonlinear Galerkin's method.

  ◇ Computation of optimal location of actuators and sensors through minimization of a cost that includes penalty on the closed-loop response and the control effort.

  ◇ Nonlinear output feedback controller design using geometric methods.

- Illustration of theoretical results through computer simulation of the closed-loop system using a high-order discretization of the KSE.

KURAMOTO-SIVASHINSKY EQUATION

- Kuramoto-Sivashinsky equation with distributed control:

$$\frac{\partial U}{\partial t} = -\nu \frac{\partial^4 U}{\partial z^4} - \frac{\partial^2 U}{\partial z^2} - U\frac{\partial U}{\partial z} + \sum_{i=1}^{l} b_i u_i(t)$$

- Boundary conditions: $\frac{\partial^j U}{\partial z^j}(-\pi, t) = \frac{\partial^j U}{\partial z^j}(+\pi, t), \quad j = 0, \ldots, 3$

- Representation in Hilbert space:

$$\dot{x} = \mathcal{A}x + \mathcal{B}u + f(x), \;\; x(0) = x_0$$
$$y_m = kx$$

$\mathcal{A}$: linear operator.

$f(x)$: nonlinear function.

EIGENSPECTRUM / OPEN-LOOP DYNAMICS

- Eigenvalue problem:

$$\mathcal{A}\phi_n = -\nu\frac{\partial^4 \phi_n}{\partial z^4} - \frac{\partial^2 \phi_n}{\partial z^2} = \lambda_n \phi_n, \quad n = 1, \dots, \infty$$

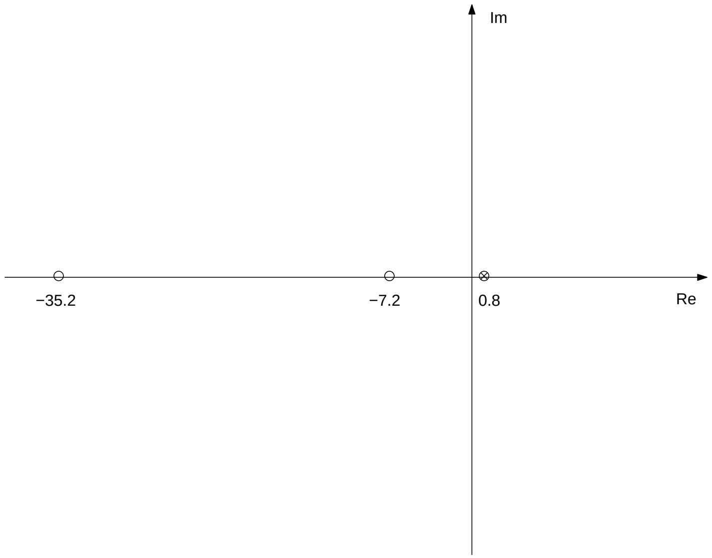
$$\frac{\partial^j \phi_n}{\partial z^j}(-\pi) = \frac{\partial^j \phi_n}{\partial z^j}(+\pi), \quad j = 0, \dots, 3$$

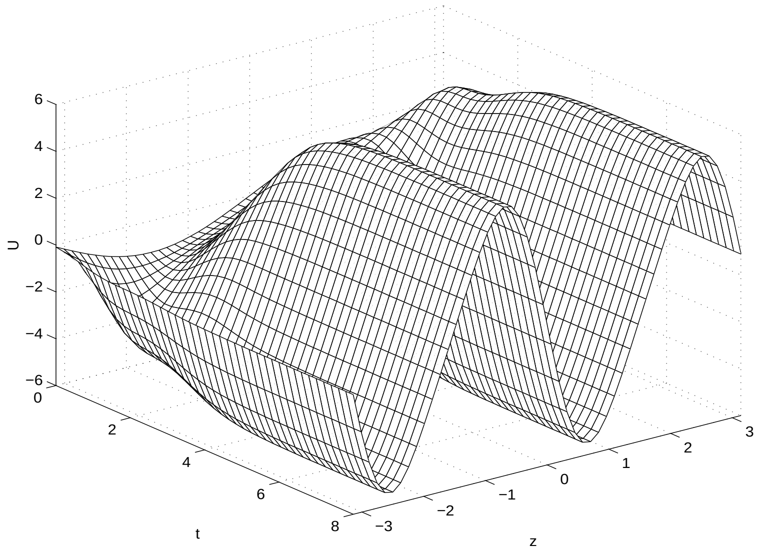
- Eigenvalues: $\lambda_n = -\nu n^4 + n^2$, $n = 1, \dots, \infty$:

Structure of Eigenspectrum

Open-loop profile of $U(z,t)$

$\nu = 0.2$





$\nu = 0.2$

GALERKIN'S METHOD

- $\mathcal{H}_s = span\{\phi_1, \phi_2, \ldots, \phi_m\}$, $\mathcal{H}_f = span\{\phi_{m+1}, \phi_{m+2}, \ldots,\}$.

$$x_s(t) = P_s x(t), \quad x_f(t) = P_f x(t)$$

P_s, P_f: orthogonal projection operators.

- Set of infinite ODEs.

$$\begin{aligned} \frac{dx_s}{dt} &= \mathcal{A}_s x_s + b_s u + f_s(x_s, x_f) \\ \frac{\partial x_f}{\partial t} &= \mathcal{A}_f x_f + b_f u + f_f(x_s, x_f) \end{aligned}$$

- Finite-set of ODEs.

$$\frac{d\tilde{x}_s}{dt} = \mathcal{A}_s \tilde{x}_s + \mathcal{B}_s u + f_s(\tilde{x}_s, 0)$$

- Order reduction using Galerkin's method and approximate inertial manifolds is also possible.

NONLINEAR CONTROL DESIGN

$$\frac{d\tilde{x}_s}{dt} = \mathcal{A}_s\tilde{x}_s + \mathcal{B}_s u + f_s(\tilde{x}_s, 0)$$

- Assumption: $l = m$ (i.e. number of manipulated inputs is equal to the number of slow modes) and $\mathcal{B}_s$ is invertible.

- Nonlinear state feedback controller:

$$u = \mathcal{B}_s^{-1}\left((\Lambda_s - \mathcal{A}_s)\tilde{x}_s - f_s(\tilde{x}_s, 0)\right)$$

Λ_s is a stable matrix.

- Closed-loop ODE system:

$$\dot{\tilde{x}}_s = \Lambda_s\tilde{x}_s$$

- Response depends on Λ_s and $x_s(0)$, but not on actuator locations:

$$\tilde{x}_s = e^{\Lambda_s t}x_s(0)$$

OPTIMAL POINT ACTUATOR PLACEMENT

- Performance criterion (sum is over a set of m linearly independent $x_s^i(0)$):

$$\hat{J}_s = \frac{1}{m}\sum_{i=1}^{m}\int_0^{\infty}((\tilde{x}_s(x_s^i(0),t),Q_s\tilde{x}_s(x_s^i(0),t)) + u^T(\tilde{x}_s(x_s^i(0),t),z_a)Ru(\tilde{x}_s(x_s^i(0),t),z_a))dt$$

- However, system's response does not depend on the actuator locations:

$$\hat{J}_{xs} = \frac{1}{m}\sum_{i=1}^{m}\int_0^{\infty}(\tilde{x}_s(x_s^i(0),t),Q_s\tilde{x}_s(x_s^i(0),t))dt$$

- Performance criterion reduces to:

$$\hat{J}_{us} = \frac{1}{m}\sum_{i=1}^{m}\int_0^{\infty}u^T(\tilde{x}_s(x_s^i(0),t),z_a)Ru(\tilde{x}_s(x_s^i(0),t),z_a)dt$$

- Computation of optimal actuator locations:

$$\begin{bmatrix}\frac{\partial \hat{J}_{us}}{\partial z_{a1}} & \frac{\partial \hat{J}_{us}}{\partial z_{a2}} & \cdots & \frac{\partial \hat{J}_{us}}{\partial z_{al}}\end{bmatrix}^T = \begin{bmatrix}0 & 0 & \cdots & 0\end{bmatrix}^T, \quad \nabla_{z_a z_a}\hat{J}_{us}(z_{am}) > 0$$

- Near-optimal solution for distributed system as $\epsilon = \frac{|\lambda_1|}{|\lambda_{m+1}|} \rightarrow 0$.

OPTIMAL LOCATION OF POINT SENSORS

$$\begin{aligned}\frac{d\tilde{x}_s}{dt} &= \mathcal{A}_s\tilde{x}_s + \mathcal{B}_s u + f_s(\tilde{x}_s, 0)\\ \tilde{y_m} &= \mathcal{S}\tilde{x}_s\end{aligned}$$

- Assumption: $p = m$(i.e. the number of sensors is equal to the number of slow modes) and $\mathcal{S}$ is invertible.
- Computation of estimate of $\tilde{x}_s$ from the measurements:

$$\hat{x}_s = \mathcal{S}^{-1} y_m$$

  y_m: sensor measurements, $\hat{x}_s$ estimate of $\tilde{x}_s$.

- Compute point sensor locations to minimize the estimation error in the closed-loop system:

$$\hat{J}(e) = \frac{1}{m}\sum_{i=1}^{m}\int_0^{\infty}(||x_s(x_s^i(0), t) - \hat{x}_s(x_s^i(0), t)||_2)dt$$

  x_s: slow state of infinite set of ODEs, $\qquad e(t) = ||x_s - \hat{x}_s||_2$

- Near-optimal solution for distributed system as $\epsilon \to 0$.

KURAMOTO-SIVASHINSKY EQUATION

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- Boundary conditions: $\frac{\partial^j U}{\partial z^j}(-\pi, t) = \frac{\partial^j U}{\partial z^j}(+\pi, t)\,, \quad j = 0, \ldots, 3$

- The stability of steady-state $U(z,t) = 0$ depends on ν.

CONTROLLER SYNTHESIS

Kuramoto-Sivashinsky equation, $\nu = 0.2$

- $\nu = 0.2 \rightarrow$ Two unstable eigenmodes.
- Order of ODE model used for controller design:2
  Number of control actuators:2; Number of measurement sensors:2
- Approximate two-dimensional ODE system used for controller design.

$$
\begin{bmatrix} (\dot{\tilde{x}}_{s1}, \phi_1) \\ (\dot{\tilde{x}}_{s2}, \phi_2) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} (\tilde{x}_{s1}, \phi_1) \\ (\tilde{x}_{s2}, \phi_2) \end{bmatrix}
$$

$$
+ \begin{bmatrix} \phi_1(z_{a1}) & \phi_1(z_{a2}) \\ \phi_2(z_{a1}) & \phi_2(z_{a2}) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}
$$

$$
+ \begin{bmatrix} (f_1(\tilde{x}_s, 0), \phi_1) \\ (f_2(\tilde{x}_s, 0), \phi_2) \end{bmatrix}
$$

z_{a1}, z_{a2} : location of point actuators.

CONTROLLER SYNTHESIS

- Nonlinear state feedback controller:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \phi_1(z_{a1}) & \phi_1(z_{a2}) \\ \phi_2(z_{a1}) & \phi_2(z_{a2}) \end{bmatrix}^{-1} * \left(\begin{bmatrix} -\alpha - \lambda_1 & 0 \\ 0 & -\beta - \lambda_2 \end{bmatrix} \begin{bmatrix} (\tilde{x}_{s1}, \phi_1) \\ (\tilde{x}_{s2}, \phi_2) \end{bmatrix} - \begin{bmatrix} (f_1(\tilde{x}_s, 0), \phi_1) \\ (f_2(\tilde{x}_s, 0), \phi_2) \end{bmatrix} \right)$$

α and β are positive real numbers.

- The structure of the closed-loop finite-dimensional system under the feedback control :

$$\begin{bmatrix} (\dot{\tilde{x}}_{s1}, \phi_1) \\ (\dot{\tilde{x}}_{s2}, \phi_2) \end{bmatrix} = \begin{bmatrix} -\alpha & 0 \\ 0 & -\beta \end{bmatrix} \begin{bmatrix} (\tilde{x}_{s1}, \phi_1) \\ (\tilde{x}_{s2}, \phi_2) \end{bmatrix}$$

which is independent of the actuator locations.

OPTIMAL LOCATION OF ACTUATORS

- Cost for optimal location of control actuators:

$$\hat{J}_s = \frac{1}{2}\sum_{i=1}^{2}\int_0^{\infty}((\tilde{x}_s^T(x_s^i(0),t),Q_s\tilde{x}_s(x_s^i(0),t)) + u^T(\tilde{x}_s(x_s^i(0),t),z_a)Ru(\tilde{x}_s(x_s^i(0),t),z_a))dt$$

$$Q_s = R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ x_s^1(0) = [\ \phi_1 \quad 0\ ],\quad x_s^2(0) = [\ 0 \quad \phi_2\ ].$$

Optimal actuator locations: $z_{a1} = 0.31\pi$ and $z_{a2} = 0.69\pi$.

OPTIMAL LOCATION OF SENSORS

- Cost for optimal location of measurement sensors:

$$\hat{J}(e) = \frac{1}{2} \sum_{i=1}^{2} \int_0^\infty (||x_s(x_s^i(0), t) - \hat{x}_s(x_s^i(0), t)||_2) dt$$

where x_s can be obtained from the simulation of the high order system, and $\hat{x}_s$ is calculated from the output:

$$\begin{bmatrix} \hat{x}_{s1} \\ \hat{x}_{s2} \end{bmatrix} = \begin{bmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{bmatrix} \begin{bmatrix} \phi_1(z_{s1}) & \phi_1(z_{s2}) \\ \phi_2(z_{s1}) & \phi_2(z_{s2}) \end{bmatrix}^{-1} \begin{bmatrix} y_{m1}(z_{s1}, t) \\ y_{m2}(z_{s2}, t) \end{bmatrix}$$

Optimal sensor locations: $z_{s1} = 0.35\pi$ and $z_{s2} = 0.64\pi$.

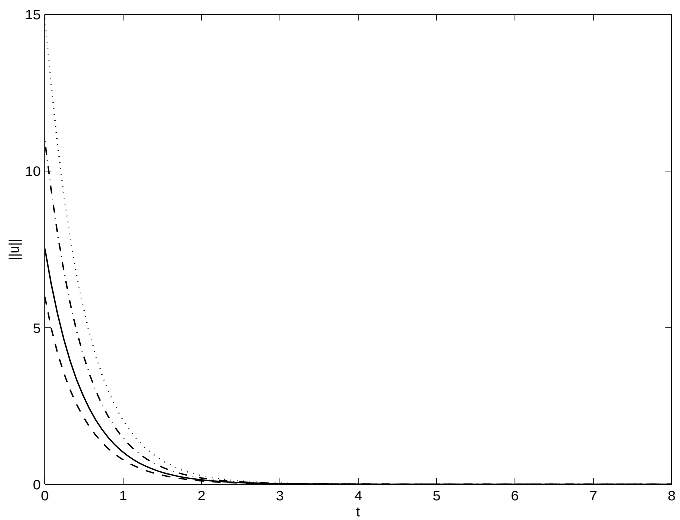
SIMULATION RESULTS

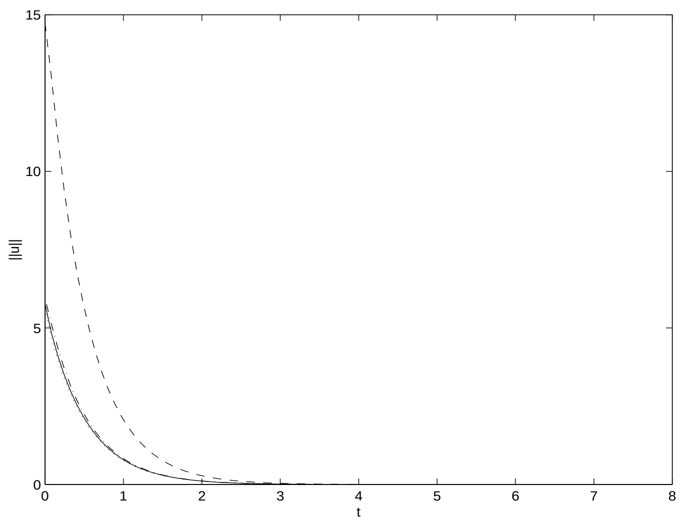
- State feedback control with two actuators.

Case	Actuator locations	$\hat{J}_s$
Optimal	$.31\pi, .69\pi$	3.3705
2	$.30\pi, .80\pi$	4.1536
3	$.40\pi, .60\pi$	5.2238
4	$.20\pi, .80\pi$	5.1089

- Closed-loop norm of $||u||$ for the optimal case (solid line), case 2 (long-dashed line), case 3 (short-dashed line), and case 4 (dotted line).

  Left plot: $x_s(0) = [\ \phi_1 \quad 0\ ]$. Right plot: $x_s(0) = [\ 0 \quad \phi_2\ ]$.





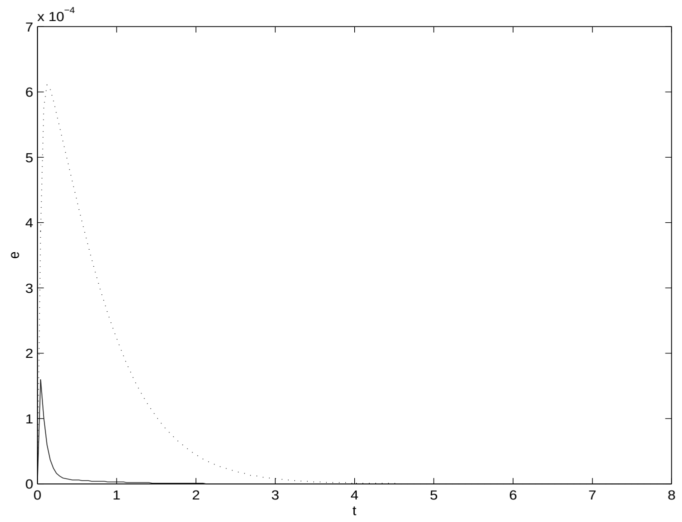
SIMULATION RESULTS

- Output feedback control with two sensors (optimal actuator locations).

Case	Sensor locations	$\hat{J}(e)$	$\hat{J}_s$
Optimal	$.35\pi, .64\pi$	2.86e-4	3.8748
2	$.20\pi, .80\pi$	1.841e-3	3.8830
3	$.20\pi, .50\pi$	5.020e-3	3.9273

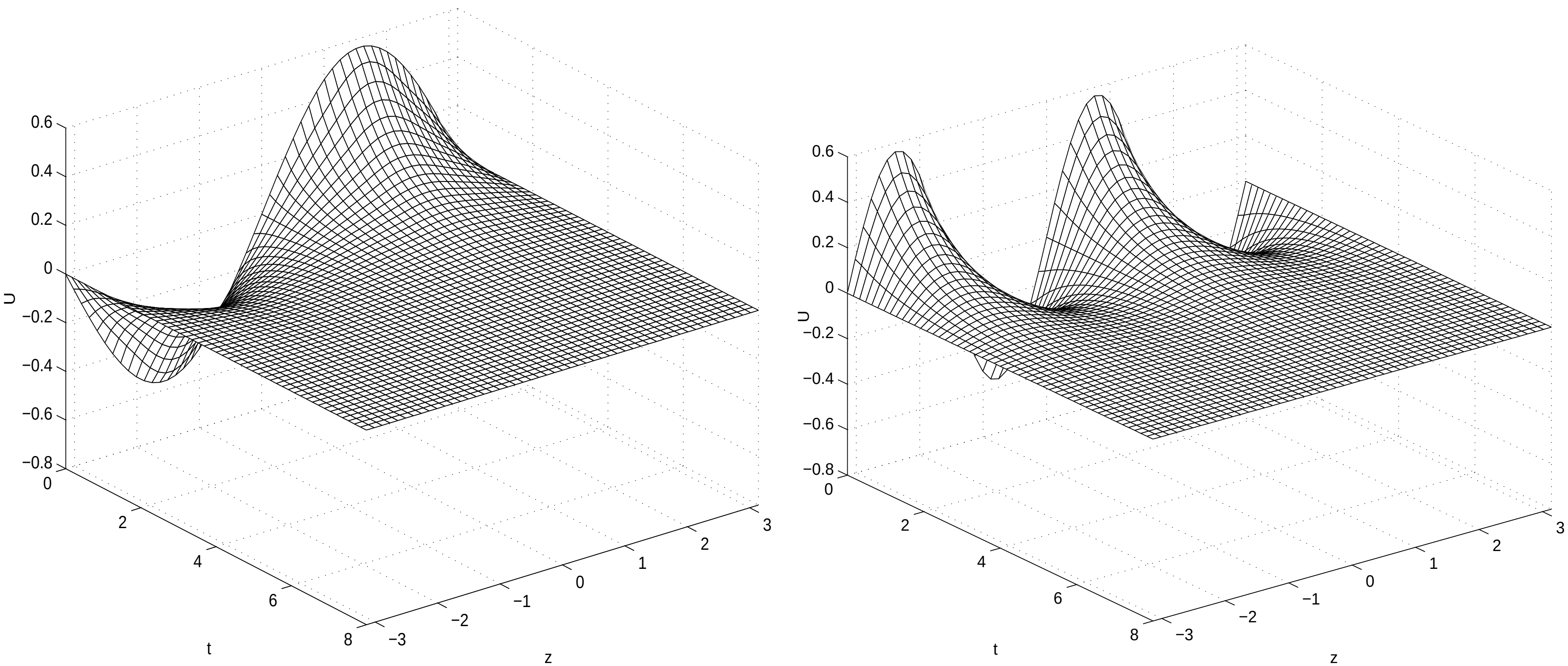
- Closed-loop norm of the closed-loop estimation error $\|e\|$ versus time, for the optimal actuator/sensor locations.

  Dotted line: $x_s(0) = [\ \phi_1 \quad 0\ ]$. Solid line: $x_s(0) = [\ 0 \quad \phi_2\ ]$.



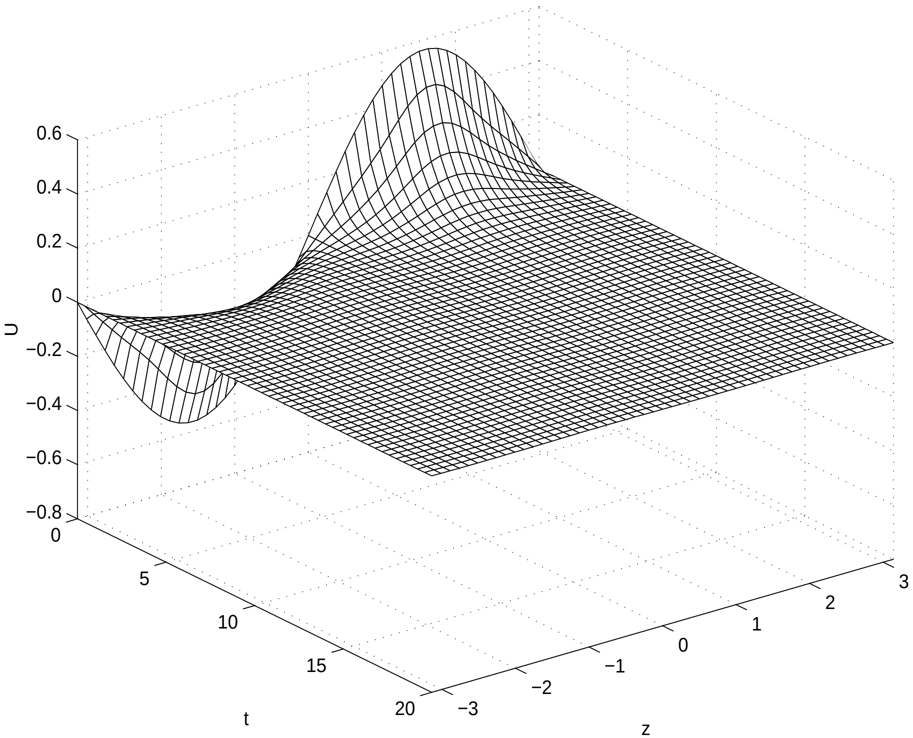
SIMULATION RESULTS

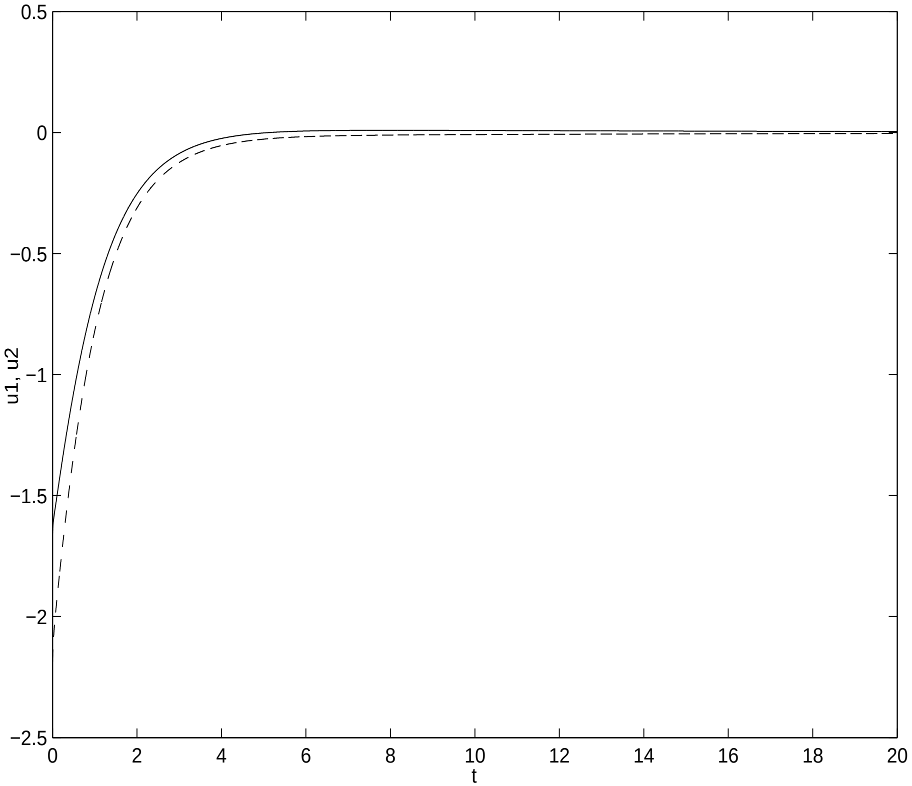
- Profiles of the evolution of U under output feedback control, for optimal actuator/sensor locations, for $x_s(0) = [\ \phi_1 \quad 0\ ]$ (left figure), and $x_s(0) = [\ 0 \quad \phi_2\ ]$ (right figure).



SIMULATION RESULTS

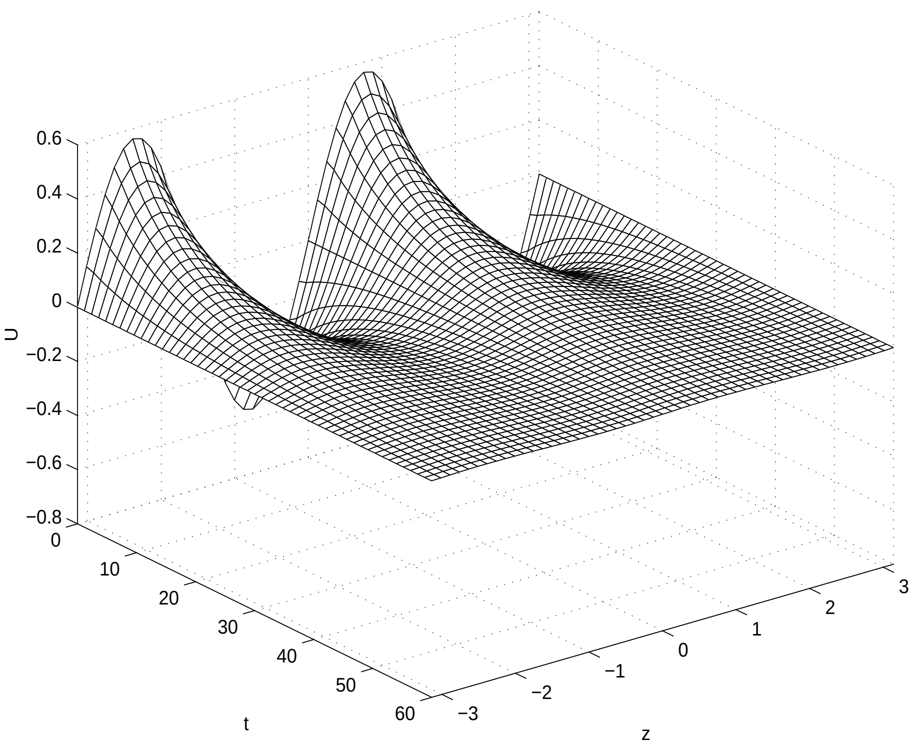
- Profiles of KSE under output feedback control, for the optimal actuator/sensor locations, for $x_s(0) = [\ \phi_1 \quad 0\ ]$ and uncertainty in ν. Manipulated input profiles - Solid line u_1 and dashed line u_2

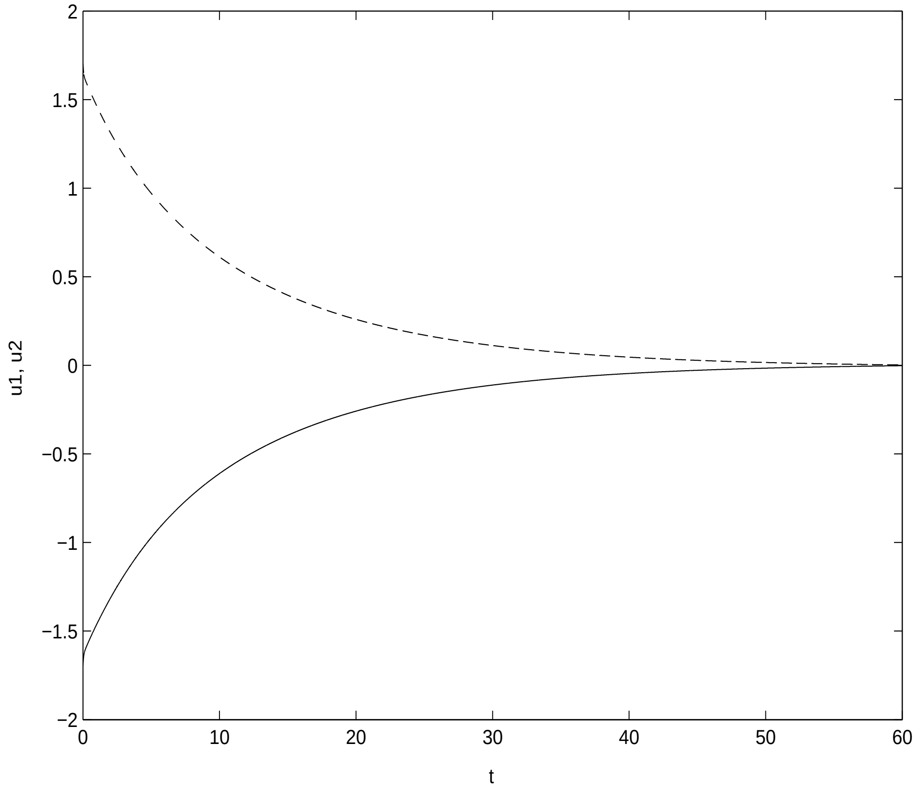




SIMULATION RESULTS

- Profiles of KSE under output feedback control, for the optimal actuator/sensor locations, for $x_s(0) = [\ 0 \quad \phi_2\ ]$ and uncertainty in ν. Manipulated input profiles - Solid line u_1 and dashed line u_2





CONCLUSIONS

- Optimal actuator/sensor placement for nonlinear control of the KSE.
  - ◇ Order reduction using Galerkin's method
  - ◇ Nonlinear output feedback controller design using geometric methods.
  - ◇ Computation of optimal actuator/sensor location through minimizing a cost that includes penalty on the close-loop response and control effort.

ACKNOWLEDGMENT

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