# HYBRID PREDICTIVE OUTPUT FEEDBACK STABILIZATION OF CONSTRAINED LINEAR SYSTEMS

**Prashant Mhaskar, Nael H. El-Farra**
**& Panagiotis D. Christofides**

**Department of Chemical Engineering**
**University of California, Los Angeles**

ADCHEM 2003
Hongkong, China
January 11-14, 2004

## INTRODUCTION

- **Input constraints:**
  - ◇ Finite capacity of control actuators
  - ◇ Impose fundamental limitations on initial conditions
- **Main issues for an effective control policy:**
  - ◇ Synthesis of stabilizing feedback laws
  - ◇ Explicit characterization of set of admissible initial conditions
- **Direct methods for control with constraints:**
  - ◇ Bounded control
    - ⋆ Constraint handling via explicit characterization of stability region
  - ◇ Model predictive control
    - ⋆ Constraint handling within open-loop optimal control setting
    - ⋆ Successful applications in industry

# LINEAR SYSTEMS WITH INPUT CONSTRAINTS

- **State space description:**

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \\ u(t) &\in \mathcal{U} \end{aligned}$$

◇ $x(t) \in \mathbb{R}^n$ : state vector

◇ $u(t) \in \mathbb{R}^m$ : control input

◇ $y(t) \in \mathbb{R}^k$ : output vector

◇ $(A, B)$ : controllable pair

◇ $(C, A)$ : observable pair

◇ $0 \in \text{int.}\,\mathcal{U}$ : compact & convex

- **Finite parameterization of controls:**

◇ $U_\Delta$ : piecewise constant functions (with period $\Delta$ & values in $\mathcal{U}$)

◇ $u(\cdot) \in U_\Delta$ characterized by sequence:

$$\{u[k]\} : \; u(t) = u(k\Delta), \; \forall\, t \in [k\Delta, (k+1)\Delta]$$

- **Stabilization of origin under constraints**

# MODEL PREDICTIVE CONTROL

- **Control problem formulation**

  ⋆ Finite-horizon optimal control:

$$P(x,t) \quad : \quad \min\{J(x,t,u(\cdot))|\ u(\cdot) \in U_\Delta\}$$

  ⋆ Performance index:

$$J(x,t,u(\cdot)) \quad = \quad F(x(t+T)) + \int_t^{t+T} \left[\|x^u(s;x,t)\|_Q^2 + \|u(s)\|_R^2\right] ds$$

◇ $\|\cdot\|_Q$ : weighted norm

◇ $T$ : horizon length

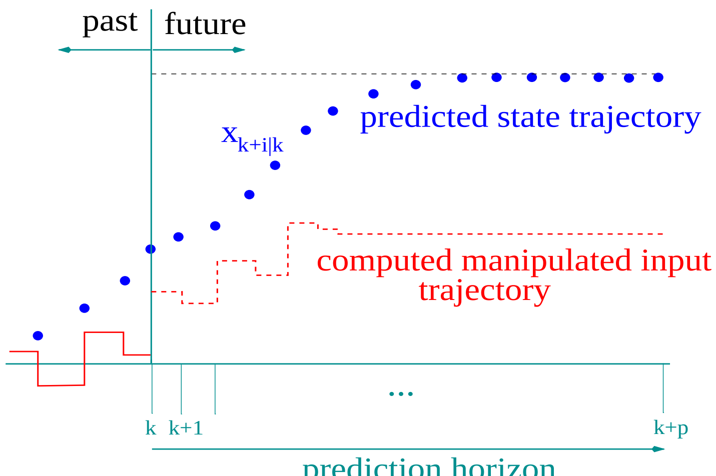
◇ $Q,\ R\ > 0$ : penalty weights

◇ $F(\cdot)$ : terminal penalty

  ⋆ Implicit feedback law

$$M(x) \quad = \quad u^0(t;x,t)$$

"repeated on-line optimization"



# MODEL PREDICTIVE CONTROL

- **Formulations for closed-loop stability:**

  (Mayne et al, Automatica, 2000)

  ◇ Adjusting horizon length, terminal penalty, weights, etc.

  ◇ Imposing stability constraints on optimization:

  ⋆ Terminal equality constraints: $x(t+T) = 0$

  ⋆ Terminal inequality constraints: $x(t+T) \in W$

  ⋆ Control Lyapunov functions: $V(x(t+T)) < V(x(t))$

- **Issues of practical implementation:**

  ◇ Lack of (a priori) explicit characterization of stability region

  ⋆ Extensive closed-loop simulations

  ⋆ Restriction to small neighborhoods around origin

# BOUNDED LYAPUNOV-BASED CONTROL

- **Explicit bounded nonlinear control law:**

$$u = -k(x, u_{max})(L_G V)^T$$

◇ An example gain: (Lin & Sontag, 1991)

$$k(x, u_{max}) = \left( \frac{L_f V + \sqrt{(L_f V)^2 + (u_{max} \|(L_G V)^T\|)^4}}{\|(L_G V)^T\|^2 \left[1 + \sqrt{1 + (u_{max}\|(L_G V)^T\|)^2}\right]} \right)$$

$$\begin{aligned} V &= x^T P x, \;\; A^T P + PA - PBB^T P < 0 \\ L_f V &= x^T (A^T P + PA) x, \;\; L_G V = 2x^T PB \end{aligned}$$

◇ Nonlinear gain-shaping procedure:

⋆ Accounts explicity for constraints & closed-loop stability

- **Constrained closed-loop properties:**

◇ Asymptotic stability

◇ Inverse optimality

$$J = \int_0^\infty (l(x) + u^T R(x) u) dt, \;\; l(x) > 0, \;\; R(x) > 0, \;\; J_{min} = V(x(0))$$

# CHARACTERIZATION OF STABILITY PROPERTIES

$$D(u_{max}) \quad = \quad \{x \in \mathbb{R}^n : L_f V < u_{max} |(L_G V)^T|\}$$

- **Properties of inequality:**
  - ◇ Describes open unbounded region where:
    - ▷ $|u| \leq u_{max} \;\; \forall \, x \in D$
    - ▷ $\dot{V} < 0 \;\; \forall \;\; 0 \neq x \in D$
  - ◇ Captures constraint-dependence of stability region
  - ◇ $D$ not necessarily invariant

- **Region of guaranteed closed-loop stability:**

$$\Omega(u_{max}) \quad = \quad \{x \in \mathbb{R}^n : V(x) \leq c_{max}\}$$

  - ◇ Region of invariance: $x(0) \in \Omega \Longrightarrow x(t) \in \Omega \subset D \, \forall \, t \geq 0$
  - ◇ Provides larger estimate than saturated linear/nonlinear controllers

## HYBRID CONTROL: UNITING BOUNDED CONTROL & MPC

(El-Farra, Mhaskar & Christofides, Automatica, 2004)

- **Objectives:**
  - ◇ Development of a framework for uniting the two approaches:
    - ▷ Reconcile tradeoffs in stability and optimality properties
      - ⋆ Explicit characterization of constrained stability region
      - ⋆ A safety net for the implementation of MPC

- **Central idea:**

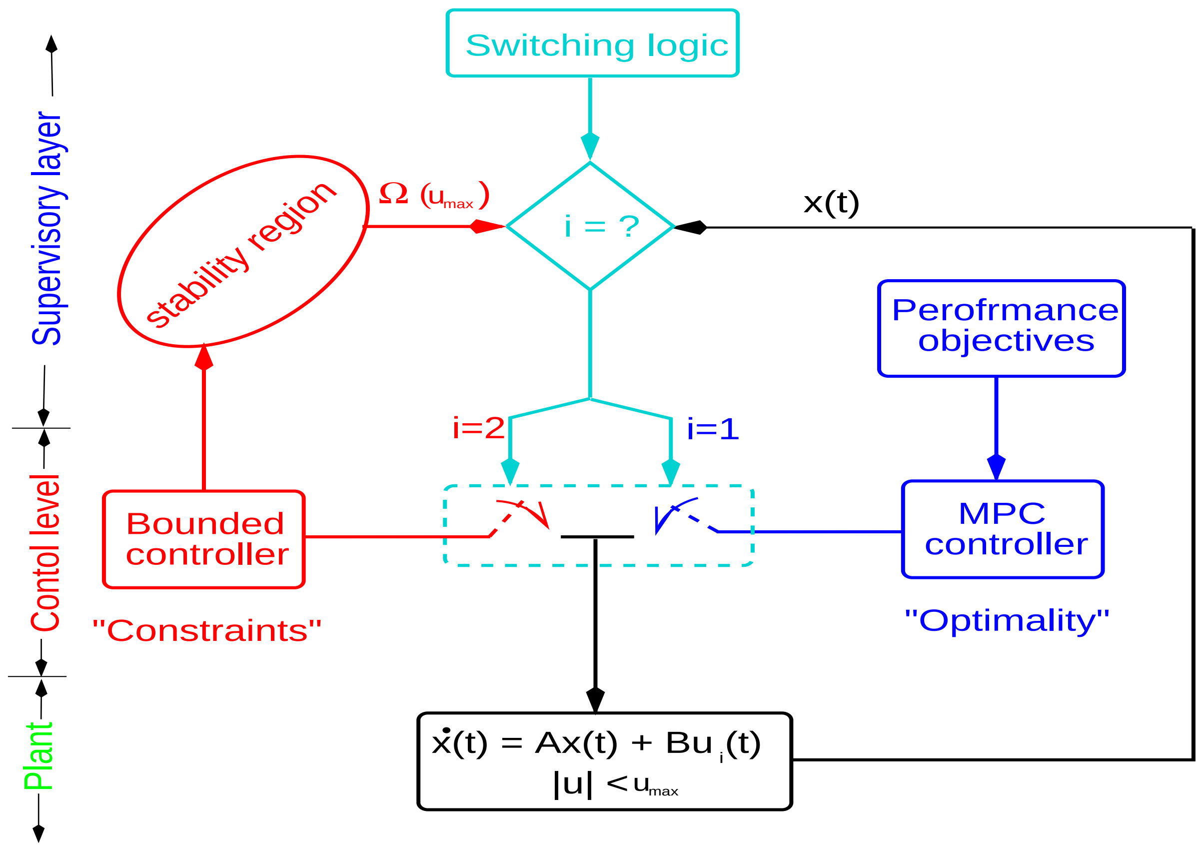
  Decoupling "optimality" & "constrained stabilizability"

  - ◇ Stability region provided by bounded controller
  - ◇ Optimal performance supplied by MPC controller

- **Approach:**
  - ◇ Switching between MPC & bounded controller

# OVERVIEW OF HYBRID CONTROL STRATEGY

"STATE FEEDBACK"




- Hierarchical control structure
  - ⋆ Plant level ⋆ Control level ⋆ Supervisory level
- Any MPC formulation can be used
  - ⋆ Switching rules may vary
- Several switching schemes possible

# CONTROLLER SWITCHING SCHEMES

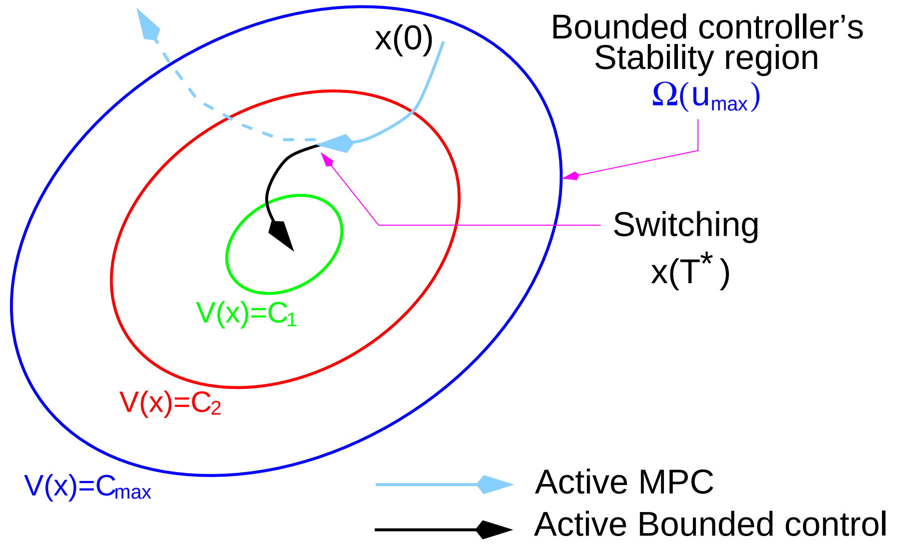
## "STATE FEEDBACK"

- **Stability-based switching:**

(e.g., Classical MPC formulations)

$$u_\sigma(x(t)) = \left\{ \begin{array}{r} M(x(t)),\ 0 \leq t < T^* \\ b(x(t)),\ \ t \geq T^* \end{array} \right\}$$

$$T^* = \inf\{T^* \geq 0 :\ V(x^M(T^*)) \geq\ 0\}$$

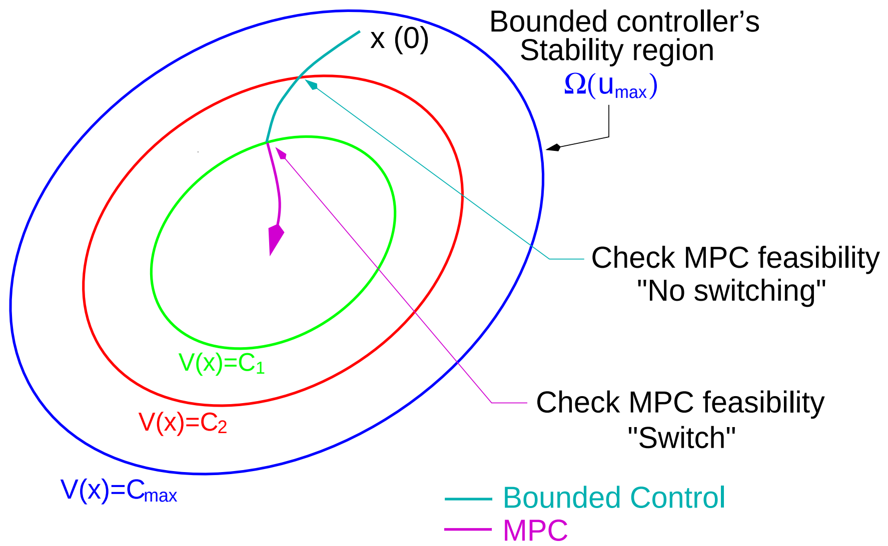



- **Feasibility-based switching:**

(Stabilizing MPC formulations)

$$u_\sigma(x(t)) = \left\{ \begin{array}{r} b(x(t)),\ 0 \leq t < T^* \\ M(x(t)),\ \ t \geq T^* \end{array} \right\}$$

⋆ $T^*$ : earliest time for which MPC yields feasible solution



# IMPLICATIONS OF SWITCHING SCHEME

- **A safety net for implementing MPC:**
  - ◇ Bounded controller provides a fall-back mechanism
  - ◇ Switched closed-loop system inherits the stability region
    - ⋆ A priori guarantees for all $x(0) \in \Omega(u_{max})$
    - ⋆ Stability independent of MPC properties (e.g., horizon length)
    - ⋆ Can reduce computational load
- **Conceptual differences from other schemes:**
  - ◇ Switching does not occur locally
  - ◇ Provides stability region explicitly
  - ◇ No switching occurs if switching rules are satisfied
    - ⋆ Only MPC is implemented $\Longrightarrow$ optimal performance recovered

# STATE ESTIMATION & OUTPUT FEEDBACK CONTROL

- **Lack of full state measurements:**
  - ◇ Inaccessibility of some process variables for measurement.
  - ◇ Estimation of states from measured outputs necessary.
- **Main objectives for output feedback controller design:**
  - ◇ To establish guaranteed stability from an explicitly characterized set of initial conditions:
    - ▷ Design technique for the state estimator.
    - ▷ Devise switching rules, based on available state measurements.
  - ◇ Controlled rate of convergence of state estimation error.
- **Approach for output feedback controller design:**
  - ◇ Relies on separation principle: combination of
    - ▷ State feedback controllers.
    - ▷ State observers.

# DESIGN OF STATE OBSERVER

- **State space description of estimator:**

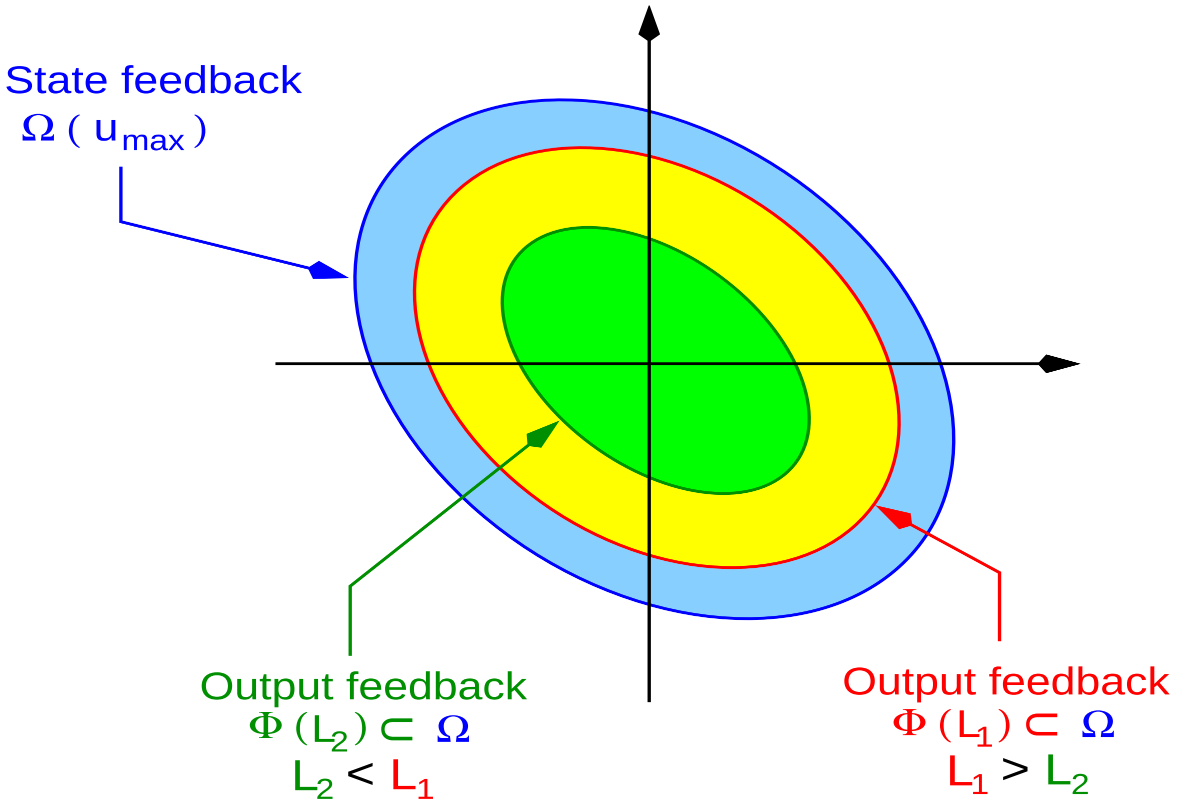
$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L(y - C\hat{x})$$

  - ◇ $\hat{x}$: state estimate.
  - ◇ $L$: Observer gain matrix.

- **Structural features:**
  - ◇ Observer pole placement:
    - ▷ Enforce desired decay of estimation error
  - ◇ Effect of observer peaking eliminated through:
    - ▷ Input saturation (indirect), OR
    - ▷ Estimate-saturation (direct)(e.g., El-Farra & Christofides, IJC, 2001).

- **Closed-loop analysis:**
  - ◇ Fall back (bounded controller) robust to a certain allowable error.
  - ◇ For a given choice of initial conditions:
    - ▷ State estimator designed to force error under the allowable error.
    - ▷ Switching laws, based on the state estimates, for "safe" MPC implementation.

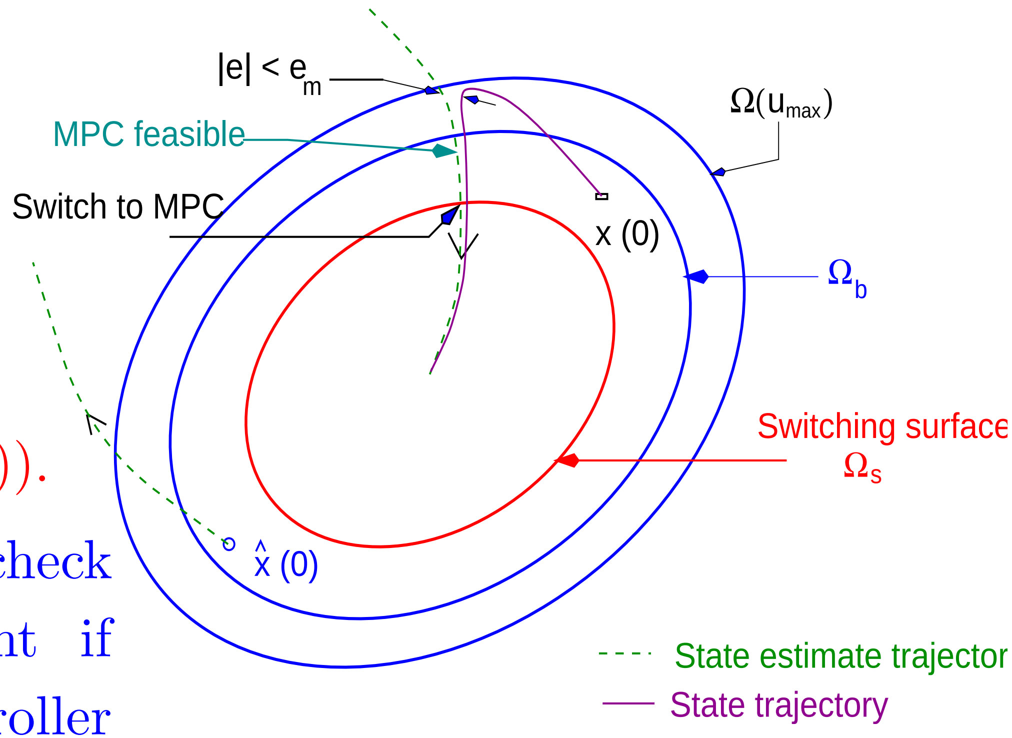
- **Practical implications:**
  - ◇ Any other estimation scheme, such as Moving Horizon Estimation (MHE), can be used.
  - ◇ Requires a transparent relationship between error decay and estimator parameters.
  - ◇ MPC implemented in a region where the fall back controller can step in any time to rescue stability.



# OUTPUT FEEDBACK IMPLEMENTATION OF SWITCHING

(Mhaskar, El-Farra, & Christofides, AIChE J., to appear)

- Bounded controller design
  ($u = -k(x)L_gV$, $\Omega(u_{max})$).
- State observer design
  (given $\Omega_b \subset \Omega$, compute $L$).
- Estimate 'safe' region
  (given $\hat{x} \in \Omega_s \Rightarrow x \in \Omega$).
- Initialize: $\hat{x}(0) \in \Omega_b$, $u(0) = b(\hat{x}(0))$.
- After $\hat{x}$ enters 'safe' region, $\Omega_s$, check feasibility of MPC & implement if $\dot{V}(\hat{x}) < 0$ else keep bounded controller active.



# SIMULATION EXAMPLE

- **State-space description:**

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}$$

$$A = \begin{bmatrix} 0.55 & 0.15 & 0.05 \\ 0.15 & 0.40 & 0.20 \\ 0.10 & 0.15 & 0.45 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

⋆ Origin unstable (three real positive eigenvalues)

⋆ Input constraints: $u_i \in [-1, 1], \;\; i = 1, 2$

⋆ Only the first and the third states are measured

# CONTROLLER DESIGN

- **Bounded controller:**
  - ◇ Lyapunov function:

$$V = x^T P x, \quad P = \begin{bmatrix} 6.5843 & 4.2389 & -3.8307 \\ 4.2389 & 3.6091 & -2.6672 \\ -3.8307 & -2.6672 & 2.8033 \end{bmatrix}$$

  - ◇ Stability region: $x^T(A^T P + PA)x \; < \; \|B^T P x\|$
- **Model predictive controller:**
  - ◇ Performance index:

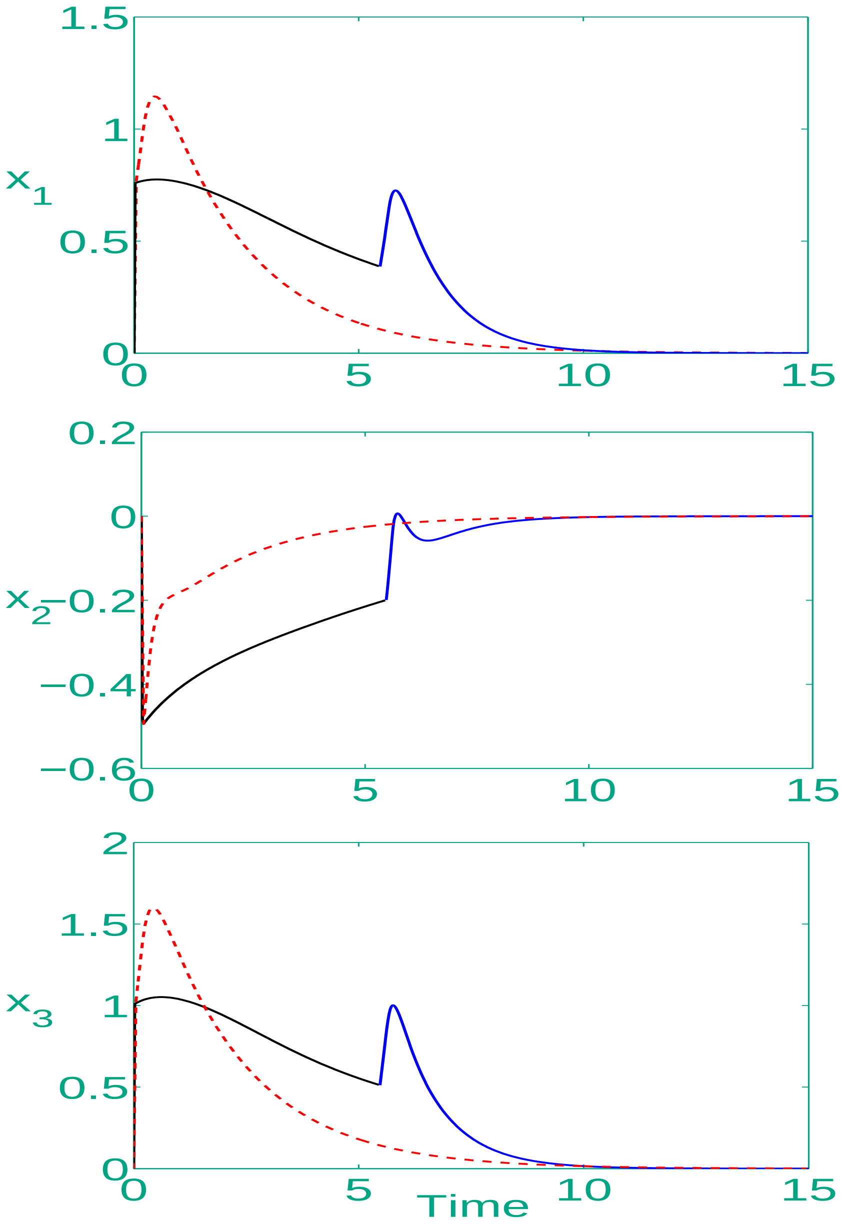
$$J = \int_t^{t+T} \left[ \|x(\tau)\|_Q^2 + \|u(\tau)\|_R^2 + \|\dot{u}(\tau)\|_S^2 \right] d\tau$$

$$Q = qI > 0, \; R = rI > 0, \; S = sI > 0$$

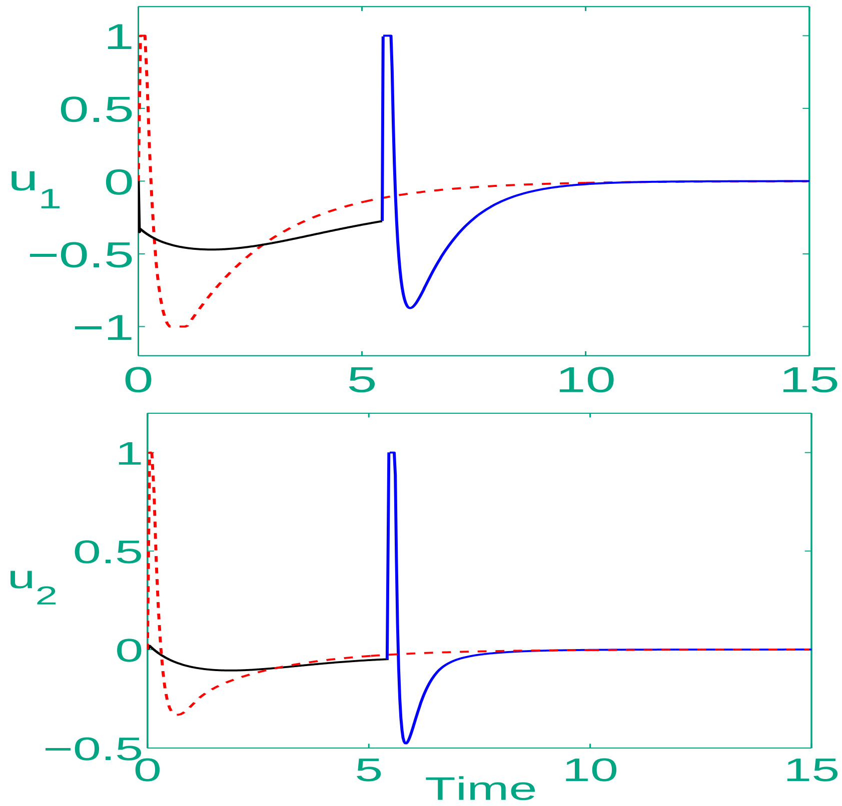
- **State observer:** $(\lambda = 500, \; a = 2)$

# CLOSED-LOOP SIMULATION RESULTS

⋆ State & estimate profiles




⋆ Input profiles




⋆ $x_0 = [0.75 \ -0.5 \ 1.0]^T \in \Omega_s$

⋆ $\hat{x}_0 = [0 \ 0 \ 0]^T \in \Omega_s$

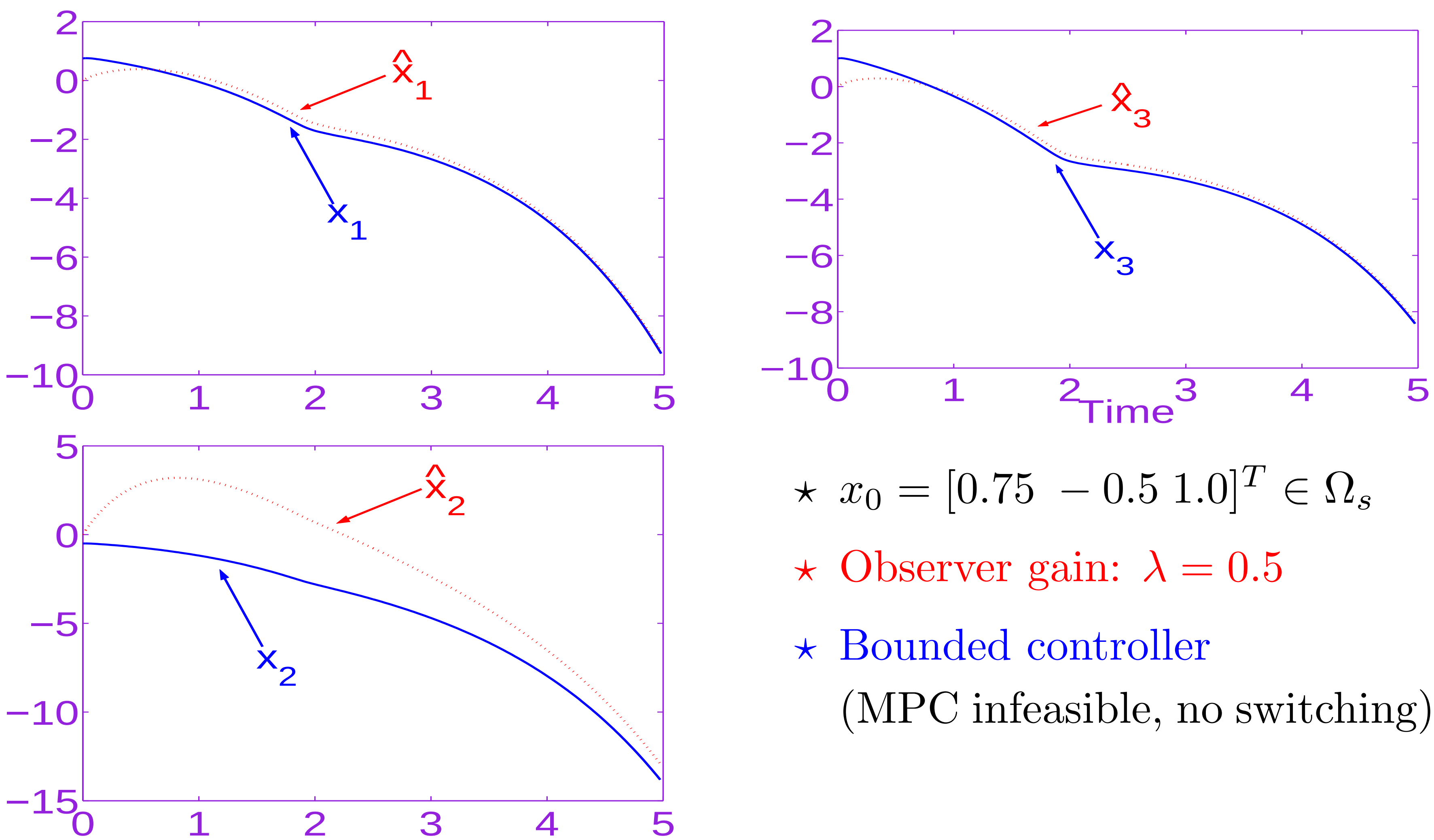
⋆ BC /MPC switching

($T = 1.5, \ t_{switch} = 5.45$)

⋆ MPC with $T = 3.5$

# CLOSED-LOOP SIMULATION RESULTS

⋆ State & estimate profiles




⋆ $x_0 = [0.75 \ -0.5 \ 1.0]^T \in \Omega_s$

⋆ Observer gain: $\lambda = 0.5$

⋆ Bounded controller

(MPC infeasible, no switching)

## CONCLUSIONS

- Output feedback stabilization of constrained linear systems
- A hybrid control structure uniting MPC & bounded control
  - ◇ Decoupling:
    - ⋆ Optimality (model predictive control)
    - ⋆ Constrained stability region (bounded control)
  - ◇ Output feedback implementation of switching strategy:
    - ⋆ State feedback controller design
    - ⋆ State observers
  - ◇ A safety net for implementation of MPC

## ACKNOWLEDGEMENT


- Financial support from NSF, CTS-0129571, & UCLA Chancellor's Fellowship is gratefully acknowledged