PREDICTIVE CONTROL OF NONLINEAR SYSTEMS WITH GUARANTEED STABILITY REGIONS

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INTRODUCTION

- **Input constraints:**
  - ◇ Finite capacity of control actuators
  - ◇ Influence stabilizability of an initial condition
- **Desired characteristics of an effective control policy:**
  - ◇ Synthesis of stabilizing feedback laws
  - ◇ Explicit characterization of set of admissible initial conditions
- **Direct methods for control with constraints:**
  - ◇ Bounded control
    - ▷ Constraint handling via explicit characterization of stability region
  - ◇ Model predictive control
    - ▷ Constraint handling within open-loop optimal control setting
    - ▷ Successful applications in industry

NONLINEAR SYSTEMS WITH INPUT CONSTRAINTS

- **State–space description:**

$$
\begin{aligned}
\dot{x}(t) &= f(x(t)) + g(x)u(t) \\
u(t) &\in \mathcal{U}
\end{aligned}
$$

◇ $x(t) \in \mathbb{R}^n$: state vector

◇ $u(t) \in \mathcal{U} \subset \mathbb{R}^m$: control input

◇ $\mathcal{U} \subset \mathbb{R}^m$: compact & convex

◇ $u = 0 \in$ interior of $\mathcal{U}$

◇ $(0,\ 0)$ an equilibrium point

- **Stabilization of origin under constraints**

MODEL PREDICTIVE CONTROL

- **Control problem formulation**
  - ◇ Finite-horizon optimal control:

$$P(x,t) \quad : \quad \min\{J(x,t,u(\cdot)) \mid u(\cdot) \in U_\Delta\}$$

  - ◇ Performance index:

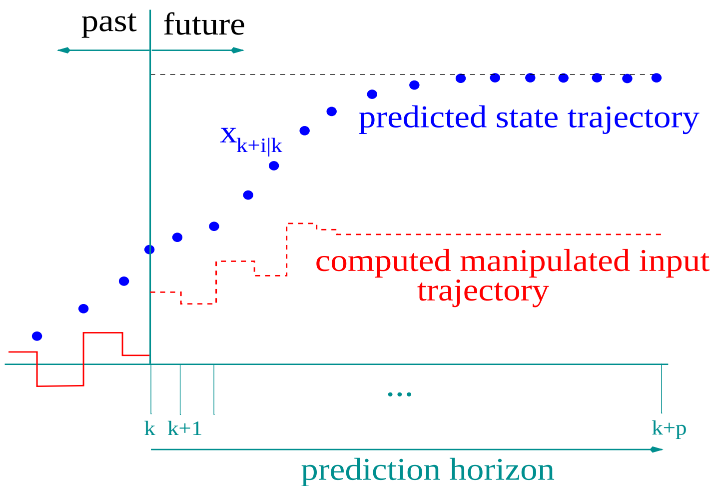
$$J(x,t,u(\cdot)) \quad = \quad F(x(t+T)) + \int_t^{t+T} \left[\|x^u(s;x,t)\|_Q^2 + \|u(s)\|_R^2\right] ds$$

- ▷ $\|\cdot\|_Q$: weighted norm
- ▷ T : horizon length
- ▷ $Q,\ R > 0$: penalty weights
- ▷ $F(\cdot)$: terminal penalty

  - ◇ Implicit feedback law

$$M(x) \quad = \quad u^0(t;x,t)$$

"repeated on-line optimization"



MODEL PREDICTIVE CONTROL

- **Formulations for closed-loop stability:**

  (Mayne et al, Automatica, 2000)

  - Adjusting horizon length, terminal penalty, weights, etc.
  - Imposing stability constraints on optimization:
    - Terminal equality constraints: $\boxed{x(t+T) = 0}$

- **Issues of practical implementation:**

  - Optimization problem non-convex
    - Possibility of multiple, local optima
    - Optimization problem hard to solve (e.g., algorithm failure)
    - Difficult to obtain solution within "reasonable" time
  - Lack of explicit characterization of stability region
    - Extensive closed-loop simulations
    - Restrict implementation to small neighborhoods

BOUNDED LYAPUNOV-BASED CONTROL

- **Explicit bounded nonlinear control law:**

$$u = -k(x, u_{max})(L_G V)^T$$

  - ◇ An example: (Lin & Sontag, 1991)

$$k(x, u_{max}) = \left(\frac{L_f V + \sqrt{(L_f V)^2 + (u_{max} \|(L_G V)^T\|)^4}}{\|(L_G V)^T\|^2 \left[1 + \sqrt{1 + (u_{max} \|(L_G V)^T\|)^2}\right]} \right)$$

  - ◇ Nonlinear gain-shaping procedure:
    - ▷ Accounts explicitly for constraints & closed-loop stability

- **Constrained closed-loop properties:**
  - ◇ Asymptotic stability
  - ◇ Inverse optimality

CHARACTERIZATION OF STABILITY PROPERTIES

$$D(u_{max}) \quad = \quad \{x \in \mathbb{R}^n : L_fV < u_{max}|(L_GV)^T|\}$$

- **Properties of inequality:**
  - ◇ Describes open unbounded region where:
    - ▷ $|u| \leq u_{max} \;\; \forall\, x \in D$
    - ▷ $\dot{V} < 0 \;\; \forall \;\; 0 \neq x \in D$
  - ◇ Captures constraint-dependence of stability region
  - ◇ D not necessarily invariant

- **Region of guaranteed closed-loop stability:**

$$\Omega(u_{max}) \quad = \quad \{x \in \mathbb{R}^n : V(x) \leq c_{max}\}$$

  - ◇ Region of invariance: $x(0) \in \Omega \Longrightarrow x(t) \in \Omega \subset D \;\forall\; t \geq 0$
  - ◇ Larger estimates using a combination of several Lyapunov functions
  - ◇ Other Lyapunov–based bounded control designs can be used

UNITING BOUNDED CONTROL AND MPC

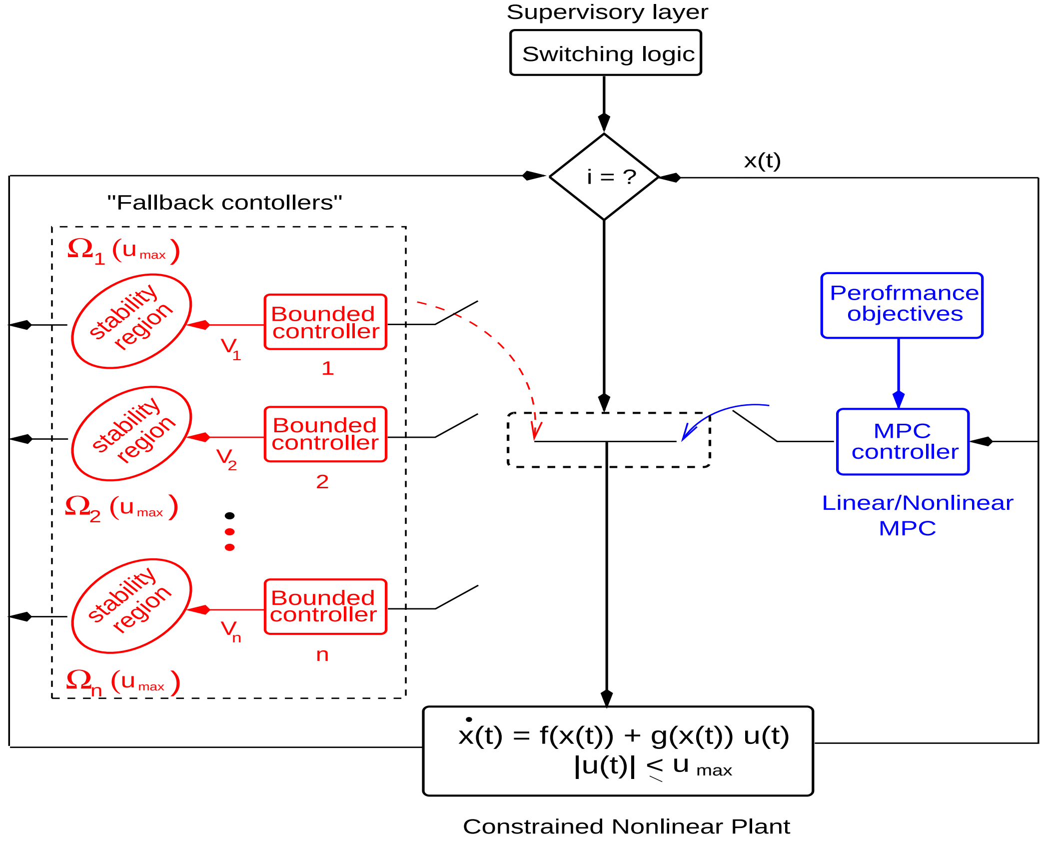
(El-Farra, Mhaskar & Christofides, Automatica, 2003; IJRNC, 2003)

- **Objectives:**
  - ◇ Development of a framework for merging the two approaches:
    - ▷ Reconcile tradeoffs in stability and optimality properties
    - ▷ Explicit characterization of constrained stability region
    - ▷ Possibility of improved performance
    - ▷ Implement computationally inexpensive MPC formulations
- **Central idea:**

Decoupling "optimality" & "constrained stabilizability"

  - ◇ Stability region provided by bounded controller
  - ◇ Optimal performance supplied by MPC controller
- **Approach:**
  - ◇ Switching between MPC & a family of bounded controllers

OVERVIEW OF HYBRID CONTROL STRATEGY



- **Hierarchical control structure**
  - ◇ Plant level ◇ Control level ◇ Supervisory level
- Overall structure **independent** of specific MPC algorithm used
  - ◇ Could use linear/nonlinear MPC with or without stability constraints

STABILITY-BASED CONTROLLER SWITCHING

- **Switching logic:**

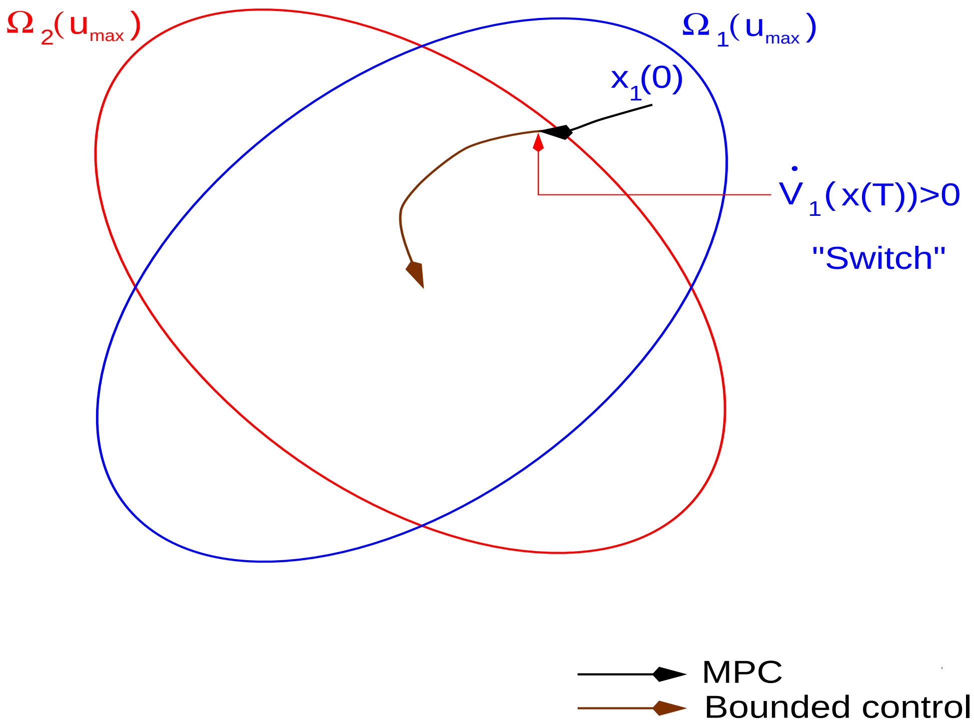
$$u_\sigma(x(t)) = \begin{cases} M(x(t)), & 0 \leq t < T^* \\ b(x(t)), & t \geq T^* \end{cases}$$

$$L_f V_k(x) + L_g V_k(x) M(x(T^*)) \geq 0$$

◇ Initially implement MPC, $x(0) \in \Omega_k(u_{max})$

◇ Monitor temporal evolution of $V_k(x^M(t))$

◇ Switch to bounded controller only if $V_k(x^M(t))$ starts to increase



ENHANCING CLOSED-LOOP PERFORMANCE

- **Switching policy:**

◇ Initialize the system in Ω_k

◇ Monitor $V_k(x)$ for which $x \in \Omega_k$

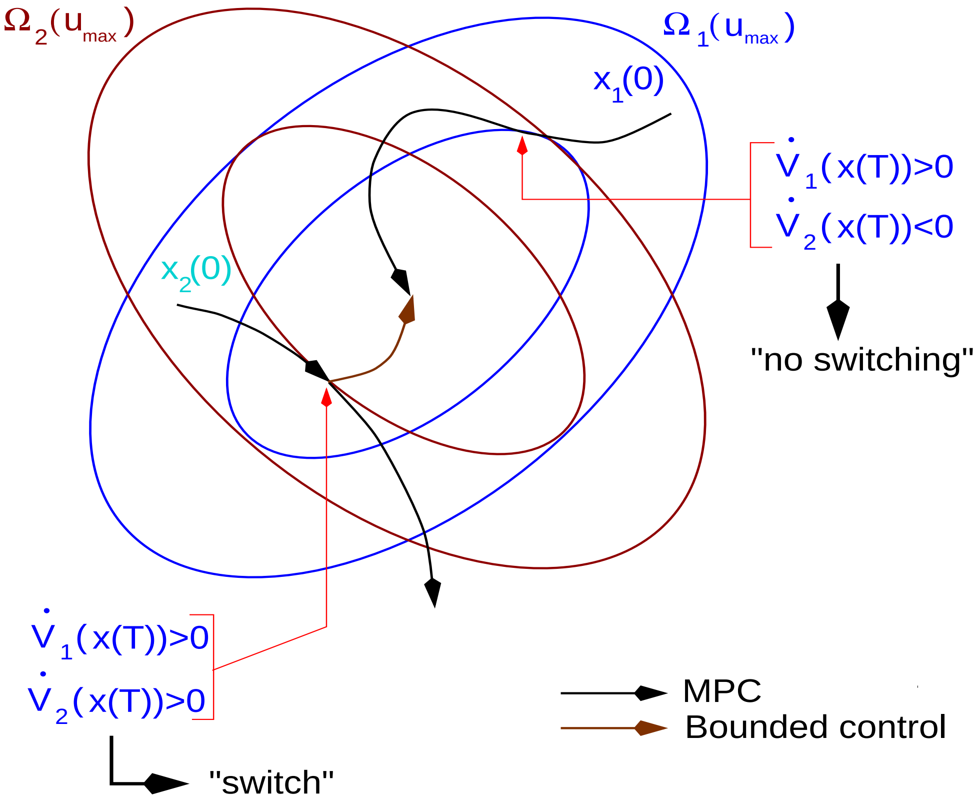
◇ Discard any V_k whose value ceases to decay

$$V_k(x(T_k))) \leq c_k^{max}$$

$$L_f V_k(x) + L_g V_k(x) M(x(t)) \geq 0$$

◇ Monitor all active V_j's for which $x \in \Omega_j$

◇ Continue MPC if $\dot{V}_j < 0$ for some active V_j. Else switch to the appropriate bounded controller



IMPLICATIONS OF SWITCHING SCHEME

- Switched closed-loop inherits bounded controller's stability region
  - ◇ A priori guarantees for all $x(0) \in \Omega(u_{max})$
- Lyapunov stability condition checked & enforced by "supervisor"
  - ◇ Reduce computational complexity of optimization
  - ◇ Scheme does not require stability of MPC within $\Omega(u_{max})$
  - ◇ Provides a safety net for implementing MPC
  - ◇ Stability independent of horizon length
- Conceptual differences from other schemes:
  - ◇ Switching does not occur locally
  - ◇ Provides stability region explicitly
  - ◇ No switching occurs if $V(x^M(t))$ decays continuously
    - ▷ Only MPC is implemented $\Longrightarrow$ optimal performance recovered

PREDICTIVE CONTROL IN INDUSTRIAL PRACTICE

- A "typical" predictive control design:
  - ◇ Nonlinear process model:

$$\dot{x} = f(x) + g(x)u$$
$$u_{min}^i \leq u_i \leq u_{max}^i$$

  - ◇ Linear representation:

$$\dot{x} = Ax + Bu$$
$$u_{min}^i \leq u_i \leq u_{max}^i$$

    - ⋆ Linearization (around desired steady-state)
    - ⋆ Model identification (e.g., through step tests)
  - ◇ Use of computationally efficient linear MPC (QP) algorithms
  - ◇ No closed-loop stability guarantees for nonlinear system
- Practical value of the hybrid control structure:
  - ◇ Provides stability guarantees through fall-back controllers
  - ◇ Entails no modifications in existing predictive controller design

APPLICATION TO A CHEMICAL REACTOR

- **State–space description:**

$$\begin{aligned}
\dot{C}_A &= \frac{F}{V}(C_{A0} - C_A) - k_0 e^{\frac{-E}{RT_R}} C_A \\
\dot{T}_R &= \frac{F}{V}(T_{A0} - T_R) + \frac{(-\Delta H)}{\rho c_p} k_0 e^{\frac{-E}{RT_R}} C_A \\
&+ \frac{UA}{\rho V c_p}(T_c - T_R)
\end{aligned}$$

◇ Multiple steady states

◇ Control objective: stabilization at the open–loop unstable equilibrium point, $(C_{As},\ T_s) = (.52,\ 398)$

◇ Manipulated input: $u = T_c \in [250,\ 500]$

APPLICATION TO A CHEMICAL REACTOR

- **Model predictive controller:**
  - ◇ Performance index:

$$J = \int_t^{t+T} \left[\|x(\tau)\|_Q^2 + \|u(\tau)\|_R^2 + \|\dot{u}(\tau)\|_S^2 \right] d\tau$$

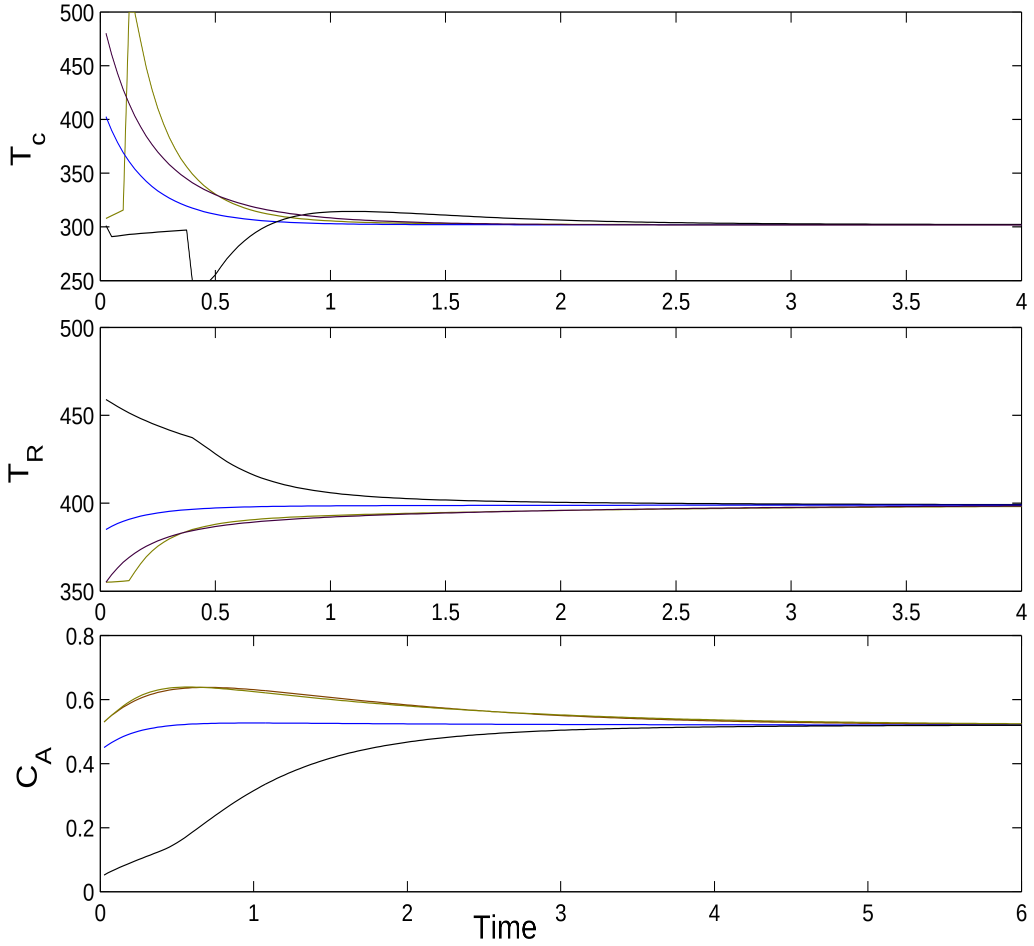
  - ◇ $Q = qI > 0, \; R = rI > 0, \; S = sI > 0$
  - ◇ Prediction model:

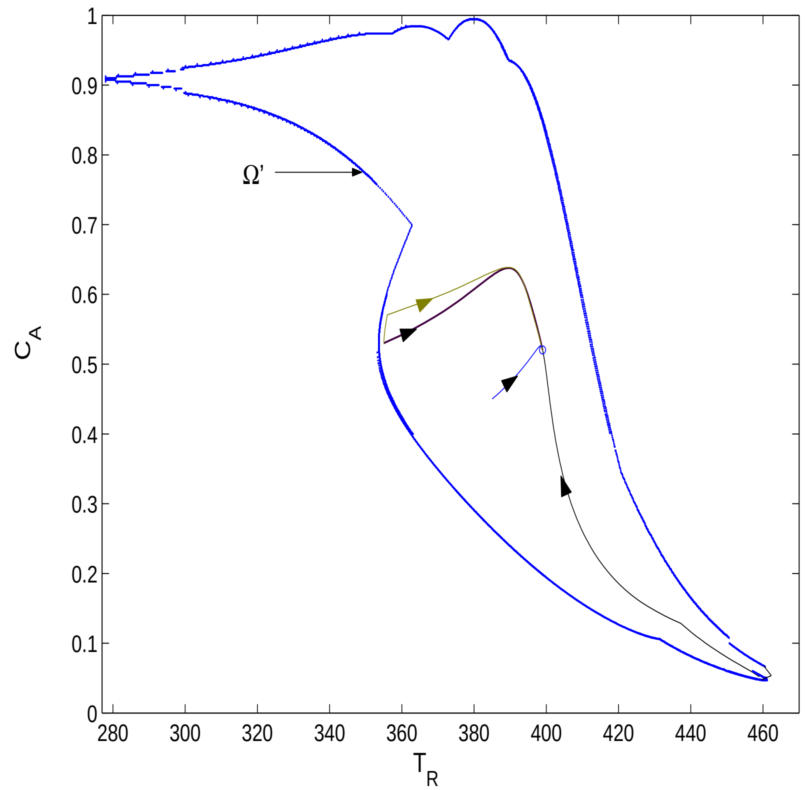
$$\dot{x} = Ax + Bu$$

    - ▷ $A, \; B$ obtained by linearizing the nonlinear model around $(C_{As}, \; T_s)$
  - ◇ Terminal equality constraint: $x(t+T) = 0$
- **Bounded controller:**
  - ◇ Bounded controller designed using a normal form representation
  - ◇ Use $V_k = \xi^T P_k \xi$

CLOSED-LOOP SIMULATION RESULTS

“Stability-based switching”





◇ Closed-loop trajectories

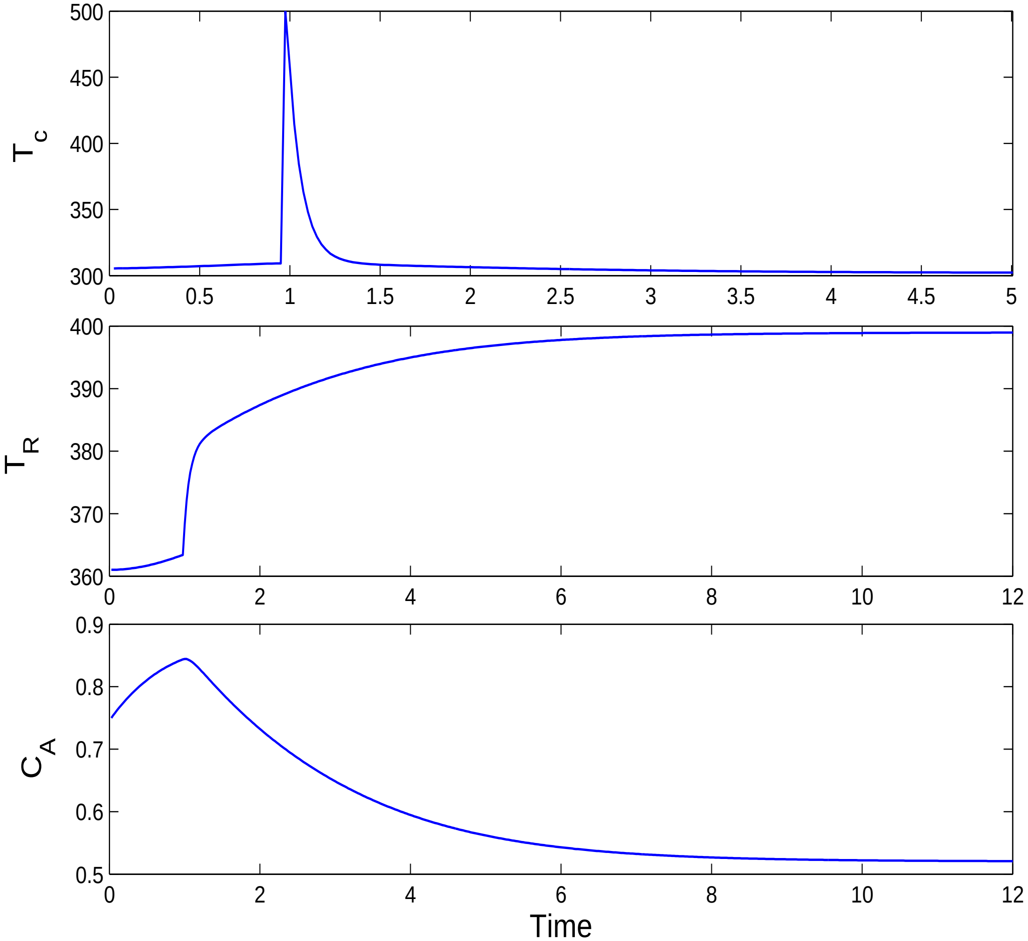
◇ Input & state profiles

◇ MPC with $T = 0.25$; MPC/BC switching $(t = 0.125)$; MPC with $T = 0.5$

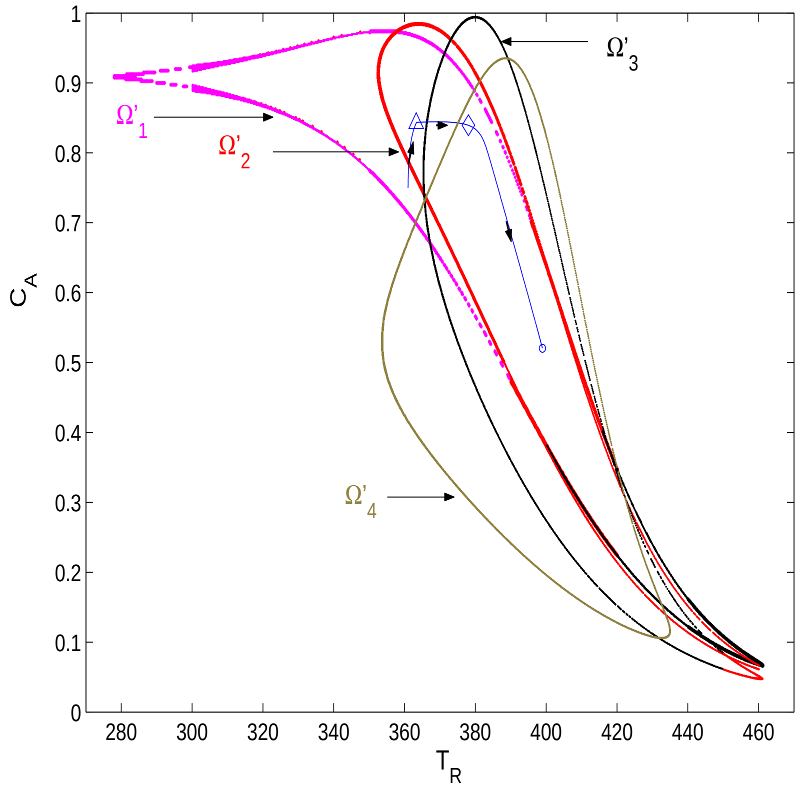
◇ MPC/BC switching $(t = 0.45)$;

CLOSED-LOOP SIMULATION RESULTS

"Performance-driven switching"



◇ Input & state profiles



◇ Closed-loop state trajectory

◇ $\dot{V}_1(t = 0.95) > 0$ $(\dot{V}_2(t = 0.95) < 0)$, $\dot{V}_2(t = 1.1) > 0$ $(\dot{V}_3(t = 1.1) < 0)$ $\Longrightarrow$

- ▷ Scheme 1: switch to bounded controller, $J = 1.81 \times 10^6$
- ▷ Scheme 2: no switching - optimal performance, $J = 1.64 \times 10^5$

APPLICATION TO A CRYSTALLIZER MOMENTS MODEL

- **State–space description:**

$$\begin{aligned}
\dot{x}_0 &= -x_0 + (1 - x_3)Dae^{\frac{-F}{y^2}} \\
\dot{x}_1 &= -x_1 + yx_0 \\
\dot{x}_2 &= -x_2 + yx_1 \\
\dot{x}_3 &= -x_3 + yx_2 \\
\dot{y} &= \frac{1 - y - (\alpha - y)yx_2}{1 - x_3} + \frac{u}{1 - x_3}
\end{aligned}$$

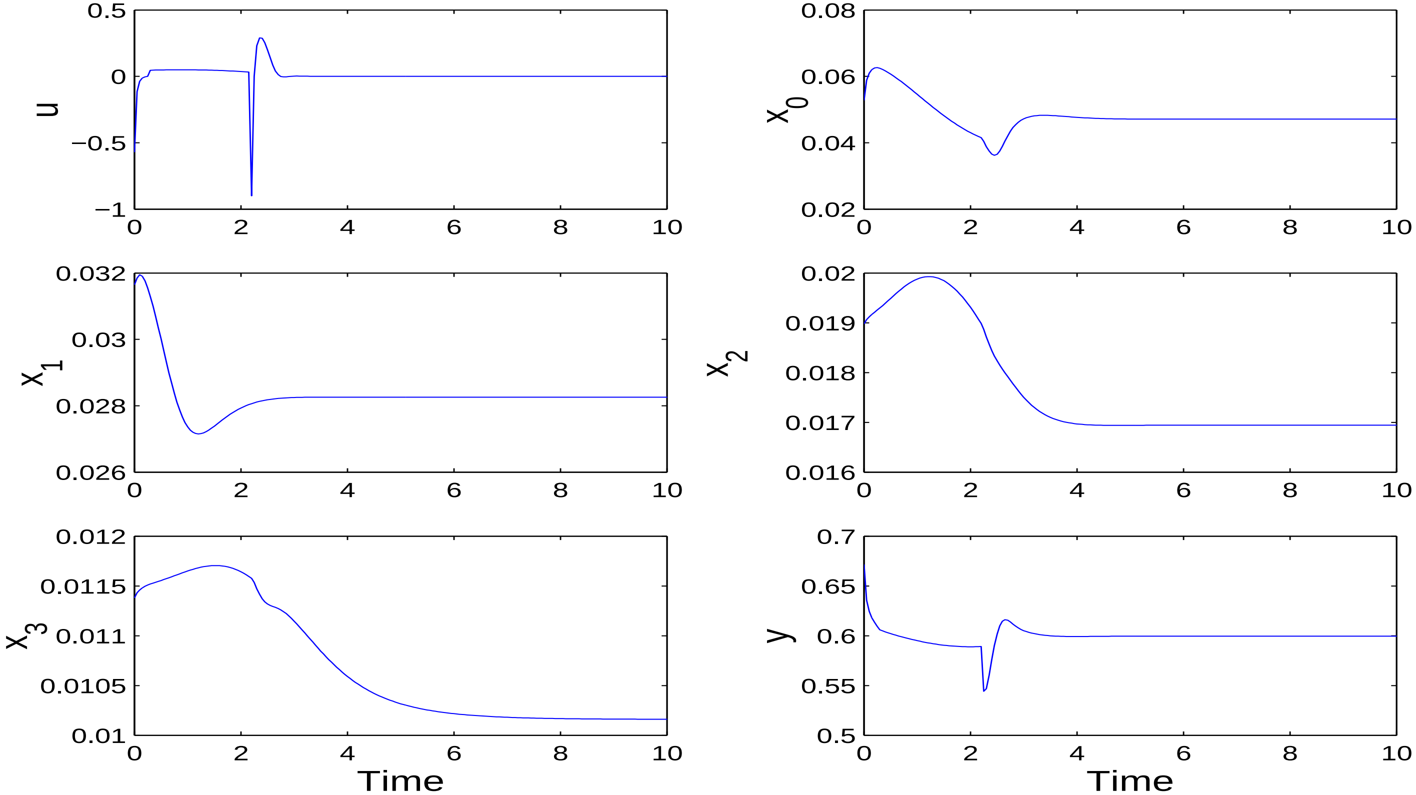
  - ◇ Unstable equilibrium point surrounded by limit cycle
  - ◇ Input constraints: $u \in [-1, 1]$

- **Bounded controller:**
  - ◇ Normal form representation:

$$\begin{aligned}
\dot{\xi} &= A\xi + bl(\xi, \eta) + b\alpha(\xi, \eta)u \\
\dot{\eta} &= \Psi(\xi, \eta)
\end{aligned}$$

CLOSED-LOOP SIMULATION RESULTS

"Stability-based switching"



◇ Input & state profiles

◇ $x(0) = [0.053\ 0.03\ 0.02\ 0.01\ 0.67\ ]^T \in \Omega_{system}(u_{max})$

◇ MPC with $T = 0.25$, switching $(t = 2.1)$, $J = 0.246$

CONCLUSIONS

- Stabilization of nonlinear systems with input constraints
- A hybrid approach uniting MPC & bounded control
  - ◇ Decoupling:
    - ▷ Optimality (model predictive control)
    - ▷ Constrained stability region (bounded control)
  - ◇ Design & implementation of switching laws:
    - ▷ Stability, performance, etc.
    - ▷ Accommodate any MPC formulation
  - ◇ Use of computationally inexpensive MPC implementations
    - ▷ Off-line explicit characterization of stability region

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